

Time boundary terms and Dirac constraints

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Abstract. Time boundary terms usually added to action principles are systematically handled in the framework of Dirac's canonical analysis. The procedure begins with the introduction of the boundary term into the integral Hamiltonian action and then the resulting action is interpreted as a Lagrangian one to which Dirac's method is applied. Once the general theory is developed, the current procedure is implemented and illustrated in various examples which are originally endowed with different types of constraints.

PACS numbers: 45.20.Jj, 45.20.-d, 45.05.+x, 03.65.Ca

1. Introduction

Time boundary terms are frequently introduced in action principles for both Lagrangian and Hamiltonian systems. On one hand, boundary terms are needed to select a complete set of commuting variables which are going to be fixed at the time boundary and thus, the choice of one of these sets is a motivation for choosing a particular boundary term or another [1, 2].

On the other hand, in gauge theories with a finite number of degrees of freedom, it is usual to deal with Hamiltonian action principles which are not fully gauge-invariant under the gauge transformation generated by their corresponding first-class constraints and in these cases it is possible to add boundary terms to these actions in such a way that the resulting action is fully gauge-invariant under the gauge transformation generated by the first-class constraints involved [1, 2, 3, 4, 5]. This is also the case for field theory [6, 7].

Independently of the motivation at hand to introduce boundary terms, it would be useful to have a general formalism or a recipe to handle them in the theoretical framework of the canonical analysis. The ideas developed in this paper are along this way of thinking. The procedure, proposed in Ref. [4] some years ago but developed here for the first time, consists in first to introduce the boundary term into the integral action and then to interpret the resulting action as a Lagrangian one to which the canonical analysis can be applied. Such a procedure has the disadvantage of enlarging the original set of variables but has the advantage of being completely systematic to handle boundary terms.

2. Theoretical framework

The starting point is a Hamiltonian system described by an action principle of the form [8, 9, 10]

$$S[q^i, p_i, u^a, v^\alpha] = \int_{\tau_1}^{\tau_2} d\tau [q^i p_i - H_E], \quad i = 1, \dots, N, \quad (1)$$

where $H_E = H_0 + u^a \gamma_a + v^\alpha \varphi_\alpha$ is the extended Hamiltonian, γ_a are first-class constraints ($a = 1, \dots, A$), φ_α are second-class constraints ($\alpha = 1, \dots, \mathcal{A}$), and H_0 is the first-class canonical Hamiltonian; u 's and v 's are their respective Lagrange multipliers. This system has $\frac{1}{2}(2N - \mathcal{A} - 2A)$ degrees of freedom. Following the ideas mentioned in the Introduction, a time boundary term is added to the action (1)

$$S[q^i, p_i, u^a, v^\alpha] = \int_{\tau_1}^{\tau_2} d\tau [q^i p_i - H_E] - B(q^i, p_i) \Big|_{\tau_1}^{\tau_2}. \quad (2)$$

Notice that, by hypothesis, B is explicitly τ -independent. The introduction of the time boundary term into the action principle yields to

$$\begin{aligned} S[q^i, p_i, u^a, v^\alpha] &= \int_{\tau_1}^{\tau_2} d\tau \left[q^i p_i - H_E - \frac{d}{d\tau} B \right] \\ &= \int_{\tau_1}^{\tau_2} d\tau \left[q^i p_i - H_E - \frac{\partial B}{\partial q^i} \dot{q}^i - \frac{\partial B}{\partial p_i} \dot{p}_i \right]. \end{aligned} \quad (3)$$

The next step is to interpret the action (3) as a Lagrangian one and so to define the momenta $(\pi_{x^\mu}) = (\pi_{q^i}, \pi_{p_i}, \pi_{u^a}, \pi_{v^\alpha})$ canonically conjugated to the configuration

variables $(x^\mu) = (q^i, p^i, u^a, v^\alpha)$, which leads to the following primary constraints

$$\begin{aligned}\phi_{q^i} &:= \pi_{q^i} - p_i + \frac{\partial B}{\partial q^i} \approx 0, & \phi_{u^a} &:= \pi_{u^a} \approx 0, \\ \phi_{p^i} &:= \pi_{p^i} + \frac{\partial B}{\partial p^i} \approx 0, & \phi_{v^\alpha} &:= \pi_{v^\alpha} \approx 0.\end{aligned}\quad (4)$$

Performing the Legendre transformation, the canonical Hamiltonian H_c is computed

$$\begin{aligned}H_c &= \pi_{q^i} \dot{q}^i + \pi_{p^i} \dot{p}^i + \pi_{u^a} \dot{u}^a + \pi_{v^\alpha} \dot{v}^\alpha - \left(\dot{q}^i p^i - H_E - \frac{\partial B}{\partial q^i} \dot{q}^i - \frac{\partial B}{\partial p^i} \dot{p}^i \right) \\ &= H_E.\end{aligned}\quad (5)$$

Therefore, the action principle is promoted to have the following Hamiltonian form

$$S[x^\mu, \pi_{x^\mu}, \lambda^\mu] := \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \pi_{x^\mu} - H_c - \lambda^\mu \phi_\mu], \quad (6)$$

where λ^μ are the corresponding Lagrange multipliers. From the variation of the action of Eq. (6), the dynamical equations

$$\begin{aligned}\dot{\pi}_{q^i} &= -\frac{\partial H_c}{\partial q^i} - \lambda^{q^j} \frac{\partial \phi_{q^j}}{\partial q^i} - \lambda^{p^j} \frac{\partial \phi_{p^j}}{\partial q^i}, \\ \dot{\pi}_{p^i} &= -\frac{\partial H_c}{\partial p^i} - \lambda^{q^j} \frac{\partial \phi_{q^j}}{\partial p^i} - \lambda^{p^j} \frac{\partial \phi_{p^j}}{\partial p^i}, \\ \dot{\pi}_{u^a} &= -\gamma_a, \\ \dot{\pi}_{v^\alpha} &= -\varphi_\alpha, \\ \dot{x}^\mu &= \lambda^\mu,\end{aligned}\quad (7)$$

together with the constraints (4) are obtained. By using the equations of motion, the evolution of the primary constraints ϕ_{q^i} and ϕ_{p^i} is computed and it is easy to check that the Lagrange multipliers associated to these constraints get fixed

$$\lambda^{q^i} = \frac{\partial H_E}{\partial p^i}, \quad \lambda^{p^i} = -\frac{\partial H_E}{\partial q^i}, \quad (8)$$

while the time evolution of ϕ_{u^a} and ϕ_{v^α} produces the following secondary constraints

$$\gamma_a \approx 0, \quad \varphi_\alpha \approx 0. \quad (9)$$

These constraints are the first- and second-class constraints of the original system. Since the evolution of the secondary constraints gives us relations among the Lagrange multipliers λ^{q^i} and λ^{p^i} which strongly vanish after plugging into them the explicit form for the Lagrange multipliers given in Eq. (8), the process ends and no more constraints arise.

In order to classify the complete set of constraints $(\phi_I) = (\phi_{q^i}, \phi_{p^i}, \phi_{u^a}, \phi_{v^\alpha}, \gamma_a, \varphi_\alpha)$, their Poisson brackets are computed and expressed in matrix form, namely,

$$(\{\phi_I, \phi_J\}) = \begin{pmatrix} \mathbf{0} & -\delta_{ij} & \mathbf{0} & \mathbf{0} & -\frac{\partial \gamma_b}{\partial q^i} & -\frac{\partial \varphi_\beta}{\partial q^i} \\ \delta_{ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\partial \gamma_b}{\partial p^i} & -\frac{\partial \varphi_\beta}{\partial p^i} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \gamma_a}{\partial q^j} & \frac{\partial \gamma_a}{\partial p^j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \varphi_\alpha}{\partial q^j} & \frac{\partial \varphi_\alpha}{\partial p^j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (10)$$

where $I, J = 1, \dots, 2(N + A + \mathcal{A})$. This matrix has a vanishing determinant which tells us that there is at least one first-class constraint. It is easy to check that the matrix (10) has $(2A + \mathcal{A})$ null vectors which implies that the constraints must be redefined. By using these null vectors, we build an inverted matrix M_I^J in order to define an equivalent set of constraints, i.e., $\tilde{\phi}_I := M_I^J \phi_J$ [10], namely

$$\tilde{\phi}_I := \begin{pmatrix} \phi_{u^a} \\ \phi_{v^\alpha} \\ \delta_a \\ \phi_{q^i} \\ \phi_{p^i} \\ \varphi_\alpha \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \delta_{ab} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta_{\alpha\beta} & \mathbf{0} & \mathbf{0} \\ \frac{\partial \gamma_a}{\partial p^j} & -\frac{\partial \gamma_a}{\partial q^j} & \mathbf{0} & \mathbf{0} & \delta_{ab} & \mathbf{0} \\ \delta_{ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta_{ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \phi_{q^j} \\ \phi_{p^j} \\ \phi_{u^b} \\ \phi_{v^\beta} \\ \gamma_b \\ \varphi_\beta \end{pmatrix}, \quad (11)$$

where $\det(M_I^J) = 1$. With this equivalent set of constraints the Poisson brackets among the constraints $\tilde{\phi}_I$ are computed again which implies that

$$\begin{aligned} \phi_{u^a} &:= \pi_{u^a} \approx 0, & \phi_{v^\alpha} &:= \pi_{v^\alpha} \approx 0, \\ \delta_a &:= \gamma_a + \frac{\partial \gamma_a}{\partial p^i} \phi_{q^i} - \frac{\partial \gamma_a}{\partial q^i} \phi_{p^i} \approx 0, \end{aligned} \quad (12)$$

are $e = (2A + \mathcal{A})$ first-class constraints, (Γ_e) , and that

$$\phi_{q^i} := \pi_{q^i} - p_i + \frac{\partial B}{\partial q^i} \approx 0, \quad \phi_{p^i} := \pi_{p^i} + \frac{\partial B}{\partial p^i} \approx 0, \quad \varphi_\alpha \approx 0, \quad (13)$$

are $\xi = (2N + \mathcal{A})$ second-class constraints, (χ_ξ) .

At this point, it is worth noticing that the second-class constraints φ_α arise from the time evolution of the first-class constraints ϕ_{v^α} . From this fact, we observe that the system analyzed here belongs to that ones which “cross the class-line in the constraint algorithm” as was pointed out in [11].

Following Dirac’s method, the first-class Hamiltonian must be built, which can be achieved by plugging into $H = H_c + \lambda_{exp}^q \chi_q$ the explicit form for the Lagrange multipliers λ_{exp}^q given in Eq. (8). Thus the extended action principle becomes

$$S[x^\mu, \pi_{x^\mu}, \lambda^e, \Lambda^\xi] := \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \pi_{x^\mu} - H - \lambda^e \Gamma_e - \Lambda^\xi \chi_\xi]. \quad (14)$$

Finally, the knowledge of the type of the constraints allows us the computation of the degrees of freedom for the system, which turns out to be $\frac{1}{2}[2N - \mathcal{A} - 2A]$ and it agrees with the original counting.

As was pointed out at the introduction one motivation in order to add boundary terms is to built action principles which are fully invariant under the gauge transformation generated by the first-class constraints, therefore we have to analyze the boundary term which arises from these transformations, such term is given by

$$M = \varepsilon^e \left(\frac{\partial \Gamma_e}{\partial \pi_{x^\mu}} \pi_{x^\mu} - \Gamma_e \right). \quad (15)$$

where ε^e are $(2A + \mathcal{A})$ gauge parameters. From the functional form of the constraints ϕ_{u^a} and ϕ_{v^α} , it is easy to see that they are linear and homogeneous in the momenta π_μ so their corresponding boundary term turns out to be zero. Therefore the unique contribution to the boundary term M is due to δ_a , i.e.,

$$M_\delta = \varepsilon^a \left(\frac{\partial \gamma_a}{\partial p_i} p_i - \gamma_a + \frac{\partial \gamma_a}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial \gamma_a}{\partial p_i} \frac{\partial B}{\partial q^i} \right). \quad (16)$$

Therefore, the extended action could be fully gauge invariant or not depending on which boundary term, B , had been chosen [4, 5].

In summary, the introduction of a time boundary term into action principle (1) and the application of the Dirac's method implies an enlargement of the phase space and the appearance of second-class constraints which contain the information about the time boundary.

3. Examples

The general theory is implemented in the following examples.

3.1. Harmonic oscillator without time boundary term

This example is relevant because it illustrates just one part of the whole procedure developed in last section: the interpretation of the original Hamiltonian action principle as a Lagrangian one. In other words, there is not a boundary term at the time boundary.

Thus, the analysis begins with the following Hamiltonian action principle for the non-relativistic one-dimensional harmonic oscillator

$$S[q, p] := \int_{\tau_1}^{\tau_2} d\tau \left[p\dot{q} - \frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2 \right], \quad (17)$$

The system has one physical degree of freedom. Following the framework described in last section, the integrand is interpreted as a Lagrangian and so from the definition of the momenta $(\pi_{x^\mu}) = (\pi_q, \pi_p)$ canonically conjugate to the coordinates $(x^\mu) = (q, p)$, the primary constraints

$$\phi_q = \pi_q - p \approx 0, \quad \text{and} \quad \phi_p = \pi_p \approx 0 \quad (18)$$

arise. With this information, the canonical Hamiltonian is computed and it turns out to be $H_c = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. Thus, the total action principle becomes

$$S[x^\mu, \pi_{x^\mu}, \lambda^\mu] = \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \pi_{x^\mu} - H_c - \lambda^\mu \phi_\mu]. \quad (19)$$

By consistency, the evolution of the constraints (18) is computed which implies that some of the Lagrange multipliers get fixed

$$\lambda^q = \frac{p}{m} \quad \text{and} \quad \lambda^p = -um\omega^2 q, \quad (20)$$

Therefore, all the Lagrange multipliers has been fixed and so there are no more constraint. A straightforward computation shows that the Poisson brackets between the constraints are $\{\phi_i, \phi_j\} = -\delta_{ij}$ where $i, j = 1, 2$ and hence (ϕ_q, ϕ_p) are second-class constraints. The first-class canonical Hamiltonian becomes $H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 q^2 + \frac{\pi_q p}{m} - m\omega^2 q \pi_p$. The counting of the number of degrees of freedom is $\frac{1}{2}(2 \times 2 - 2) = 1$, which is in agreement with the original description of the system given by the action (17).

Finally, due to there are no first-class constraints the boundary term M will not exist.

3.2. Parameterized harmonic oscillator with time boundary term

The next example is the parameterized non-relativistic one-dimensional harmonic oscillator. The example is relevant because on it is seen the role of the boundary term in the gauge invariance of the action. As starting point, we take the Hamiltonian action principle for the parameterized non-relativistic one-dimensional harmonic,

$$S[q, t, p_q, p_t; u] := \int_{\tau_1}^{\tau_2} d\tau [\dot{q}p + p_t \dot{t} - H_E], \quad (21)$$

where the extended Hamiltonian, $H_E = u\gamma$, is composed by the first-class constraint $\gamma := p_t + \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$ and the Lagrange multiplier u . On the other hand, we take the boundary term $B = B(q, t, p, p_t)|_{\tau_1}^{\tau_2}$. By applying the method, we add this boundary term to the action principle (21), namely

$$S[q, t, p_q, p_t, u] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{q}p + \dot{t}p_t - H_E - \frac{dB}{d\tau} \right]. \quad (22)$$

As next step, the integrand of (22) is interpreted as Lagrangian ones, i.e., $\mathcal{L} := \dot{q}p + \dot{t}p_t - u[p_t + \frac{1}{2m}(p^2 + m^2\omega^2 q^2)] - \frac{\partial B}{\partial q}\dot{q} - \frac{\partial B}{\partial t}\dot{t} - \frac{\partial B}{\partial p}\dot{p} - \frac{\partial B}{\partial p_t}\dot{p}_t$. The canonical analysis starts by the definition of the momenta, $\pi_{x^\mu} = (\pi_q, \pi_t, \pi_p, \pi_{p_t}, \pi_u)$, canonically conjugated to the coordinates, $x^\mu = (q, t, p_q, p_t, u)$. From these definitions it is easy to see that five primary constraints arise

$$\begin{aligned} \phi_q &:= \pi_q - p + \frac{\partial B}{\partial q} \approx 0, & \phi_t &:= \pi_t - p_t + \frac{\partial B}{\partial t} \approx 0, \\ \phi_{p_t} &:= \pi_{p_t} + \frac{\partial B}{\partial p_t} \approx 0, & \phi_p &:= \pi_p + \frac{\partial B}{\partial p} \approx 0, \\ \phi_u &:= \pi_u \approx 0. \end{aligned} \quad (23)$$

Using this information, the canonical Hamiltonian becomes $H_c = u[p_t + \frac{1}{2m}(p^2 + m^2\omega^2 q^2)]$ and the total action principle acquires the form

$$S[x^\mu, \pi_{x^\mu} \lambda^\mu] := \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \pi_{x^\mu} - H_c + \lambda^{x^\mu} \phi_{x^\mu}]. \quad (24)$$

By consistency, the primary constraints are evolved in time and from this it is straightforward to observe that four Lagrange multipliers are fixed,

$$\lambda^q = u \frac{p}{m}, \quad \lambda^t = u, \quad \lambda^p = -u m \omega^2 q, \quad \lambda^{p_t} = 0, \quad (25)$$

and that ϕ_u generates a secondary constraint

$$\Phi = p_t + \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \approx 0. \quad (26)$$

The time evolution of the secondary constraint yields a relationship among the Lagrange multipliers, which strongly vanishes after plugging into it the explicit values of the Lagrange multipliers (25), and therefore, no more constraints arise.

The Poisson brackets among the constraints are computed in order to classify them. From this, it is straightforward to obtain that $\delta_1 := \phi_u$ and

$$\begin{aligned} \delta &:= \gamma + \frac{p}{m}\phi_q + \phi_t - m\omega^2 \phi_p \\ &= \pi_t + \frac{p\pi_q}{m} - m\omega^2 q\pi_p + \frac{p}{m} \left(\frac{\partial B}{\partial q} - \frac{p}{2} \right) + m\omega^2 q \left(\frac{q}{2} - \frac{\partial B}{\partial p} \right) + \frac{\partial B}{\partial t} \end{aligned} \quad (27)$$

are first-class constraints and that the remaining ones are second-class constraints.

The first class Hamiltonian is $H = u\delta$. Moreover, the extended action principle takes the form

$$S[x^\mu, \pi_{x^\mu}, \lambda^e, \Lambda^\xi] := \int_{\tau_1}^{\tau_2} d\tau [\dot{x}^\mu \pi_{x^\mu} - H - \lambda^e \Gamma_e - \Lambda^\xi \chi_\xi]. \quad (28)$$

where $e = 1, 2$ y $\xi = 1, \dots, 4$. The last step of the procedure is to make the counting of degrees of freedom which is $\frac{1}{2}[2(5) - 4 - 2(2)] = 1$, this number agrees with the original system.

As final comment, we will consider the boundary term which arises from the gauge transformation of the action principle generated by the first-class constraints δ . The boundary term which arises from the gauge transformation generated by δ is

$$\begin{aligned} M &= \varepsilon \left(\frac{\partial \delta}{\partial \pi_\mu} \pi_\mu - \delta \right) \\ &= \varepsilon \left[\frac{p}{m} \left(\frac{p}{2} - \frac{\partial B}{\partial q} \right) - m\omega^2 \left(\frac{q}{2} - \frac{\partial B}{\partial p} \right) - \frac{\partial B}{\partial t} \right], \end{aligned} \quad (29)$$

therefore, in the general case, the extended action principle is not fully gauge invariant. Nevertheless, if the boundary term, B , is chosen to be $[2, 4]$

$$B = \frac{1}{2}qp, \quad (30)$$

the extended action principle becomes fully gauge invariant!

3.3. A system originally defined by first-class constraints only

The next example is the system described by the Hamiltonian action principle

$$S[q^i, p_i, u^i] := \int_{\tau_1}^{\tau_2} d\tau [\dot{q}^i p_i - u^i \gamma_i], \quad i = 1, 2, \dots, N, \quad (31)$$

where $\gamma_i := (p_i - q_i) \approx 0$ are N first-class constraints, u^i are Lagrange multipliers, and so the system has zero degrees of freedom.

Next, a boundary term is added to the action (31)

$$S[q^i, p_i, u^i] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{q}^i p_i - u^i \gamma_i - \frac{dB(q, p)}{d\tau} \right]. \quad (32)$$

By interpreting the integrand as a Lagrangian function, i.e., $\mathcal{L} = \dot{q}^i p_i - u^i \gamma_i - \frac{\partial B}{\partial q^i} \dot{q}^i - \frac{\partial B}{\partial p_i} \dot{p}_i$, the momenta π_{x^μ} canonically associated to the coordinates $(x^\mu) = (q^i, p_i, u^i)$ must be defined from which $3N$ primary constraints arise

$$\phi_{q^i} := \pi_{q^i} - p_i + \frac{\partial B}{\partial q^i} \approx 0, \quad \phi_{p^i} := \pi_{p^i} + \frac{\partial B}{\partial p^i} \approx 0, \quad \phi_{u^i} := \pi_{u^i} \approx 0. \quad (33)$$

The computation of the canonical Hamiltonian leads to $H_c := u^i \gamma_i$, and the total action principle becomes

$$S[x^\mu, \pi_{x^\mu}, \lambda^{x^\mu}] = \int_{\tau_1}^{\tau_2} d\tau \left[\dot{x}^\mu \pi_{x^\mu} - H_c - \lambda^{q^i} \phi_{q^i} - \lambda^{p^i} \phi_{p^i} - \lambda^{u^i} \phi_{u^i} \right]. \quad (34)$$

From the evolution of the primary constraints ϕ_{q^i} and ϕ_{p^i} , the first $2N$ Lagrange multipliers are determined

$$\lambda^{q^i} = u^i, \quad \lambda^{p^i} = u^i, \quad (35)$$

while from the evolution of ϕ_{u^i} , the constraints γ_i of the original description of the system (31) appear again, this time as secondary constraints

$$\gamma_i = p_i - q_i \approx 0, \quad (36)$$

whose evolution does not generate additional constraints. A straightforward computation shows that ϕ_{u^i} and

$$\begin{aligned} \delta_i &= \gamma_i + \phi_{q^i} - \phi_{p^i} \\ &= \pi_{q^i} - \pi_{p^i} - q^i + \frac{\partial B}{\partial q^i} - \frac{\partial B}{\partial p^i} \end{aligned} \quad (37)$$

are first-class and that ϕ_{q^i} and ϕ_{p^i} are second-class. The first-class canonical Hamiltonian becomes $H = u^i \delta_i$. The last step is to make the counting of the degrees of freedom of the system, which is $\frac{1}{2}(2 \times 3N - 2N - 2 \times 2N) = 0$, that is in agreement with the original description for the system given by the action (31).

Finally, the boundary term which arises from the gauge transformation generated by the first-class constraints acquire the form

$$M_\delta = \varepsilon \left(\frac{\partial B}{\partial p^i} - \frac{\partial B}{\partial q^i} + q^i \right). \quad (38)$$

Thus, the extended action principle will be gauge invariant or not depending on which boundary term B had be chosen.

3.4. Action for a particle on a sphere S^2

Now is considered the “free particle” restricted to move through a sphere, this system contains second-class constraints due to the restriction of the motion to the sphere. The system is described by the Hamiltonian action principle [12]

$$S[q^\alpha, p_\alpha, v^\alpha] := \int_{\tau_1}^{\tau_2} d\tau [\dot{q}^\alpha p_\alpha - H_0 - v^\alpha \varphi_\alpha], \quad \alpha = 1, \dots, 4, \quad (39)$$

with $(q^\alpha) = (x, y, z, u)$, $H_0 = \frac{|\vec{p}|^2}{2m} + \frac{u}{2}(|\vec{q}|^2 - R^2)$ is the first-class Hamiltonian and R is a constant; $\varphi_1 := p_u \approx 0$, $\varphi_2 := |\vec{q}|^2 - R^2 \approx 0$, $\varphi_3 := \vec{q} \cdot \vec{p} \approx 0$ and $\varphi_4 := \frac{|\vec{p}|^2}{m} - u|\vec{q}|^2 \approx 0$ are second-class constraints and v^α are their respective Lagrange multipliers. The system has two degrees of freedom. Next, a time boundary term is added to the action (39)

$$S[q^\alpha, p_\alpha, v^\alpha] := \int_{\tau_2}^{\tau_1} d\tau [\dot{q}^\alpha p_\alpha - H_0 - v^\alpha \varphi_\alpha - \frac{d}{d\tau} B(q^\alpha, p_\alpha)]. \quad (40)$$

By interpreting the integrand as a Lagrangian function, the momenta $(\pi_{x^\mu}) = (\pi_{q^\alpha}, \pi_{p_\alpha}, \pi_{v^\alpha})$ canonically conjugated to the coordinates $x^\mu = (q^\alpha, p^\alpha, v^\alpha)$ must be defined from which twelve primary constraints arise

$$\phi_{q^\alpha} := \pi_{q^\alpha} - p_\alpha + \frac{\partial B}{\partial q^\alpha} \approx 0, \quad \phi_{p^\alpha} := \pi_{p^\alpha} + \frac{\partial B}{\partial p_\alpha} \approx 0, \quad \phi_{v^\alpha} := \pi_{v^\alpha} \approx 0, \quad (41)$$

The computation of the canonical Hamiltonian leads to $H_c = \frac{|\vec{p}|^2}{2m} + (\frac{u}{2} + v^2)(|\vec{q}|^2 - R^2) + v^1 p_u + v^3 \vec{q} \cdot \vec{p} + v^4 (\frac{|\vec{p}|^2}{m} - u|\vec{q}|^2)$ and the total action principle becomes

$$S[x^\mu, \pi_{x^\mu}, \lambda^\mu] := \int_{\tau_1}^{\tau_2} d\tau [\dot{q}^\mu \pi_{x^\mu} - H_c - \lambda^{x^\mu} \phi_{x^\mu}]. \quad (42)$$

By consistency the constraints must be evolved and it is easy to see that the evolution of $\phi_x, \dots, \phi_{p_u}$ fix the first eight Lagrange multipliers

$$\begin{aligned}\lambda^{q^i} &= \frac{p^i}{m}(1 + 2v^4) + q^i v^3, & \lambda^{p^i} &= uq^i(2v^4 - 1) - 2q^i v^2 - p^i v^3, \\ \lambda^u &= v^1, & \lambda^{p_u} &= v^4 |\vec{x}|^2 - \frac{1}{2}(|\vec{q}|^2 - R^2),\end{aligned}\quad (43)$$

while the time evolution of ϕ_{u^α} yield the original constraints from (39), i.e.,

$$\begin{aligned}\varphi_1 &= p_u \approx 0, & \varphi_2 &= |\vec{q}|^2 - R^2 \approx 0, \\ \varphi_3 &= \vec{q} \cdot \vec{p} \approx 0, & \varphi_4 &= \frac{|\vec{p}|^2}{m} - u|\vec{q}|^2 \approx 0,\end{aligned}\quad (44)$$

whose evolution give no more constraints. A straightforward computation shows that ϕ_{v^α} are first-class and that the remaining ones are second-class. By using these information the number of physical degrees of freedom of the system is computed which turns out to be $\frac{1}{2}[2(12) - 12 - 2(4)] = 2$, which is in agreement with the original description for the system given by the action (39).

Finally, the first-class constraints ϕ_{v^α} of the system are linear and homogeneous therefore the boundary term which arise from the gauge transformation of the extended action is zero and hence the system is fully invariant no matter what boundary term had been chosen.

3.5. Action defined by a time boundary term only

This system is defined by an action principle of the form

$$S[q^i] = B(q^i) \big|_{\tau_1}^{\tau_2}, \quad i = 1, \dots, N. \quad (45)$$

By introducing the time boundary term into the integral action leads to

$$S[q^i] = \int_{\tau_1}^{\tau_2} d\tau \frac{dB}{d\tau} = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{\partial B}{\partial q^i} \dot{q}^i \right). \quad (46)$$

The next step is to interpret the action (46) as a Lagrangian one, i.e., the q 's are interpreted as configuration variables and so Dirac's method calls for the definition of their canonically conjugate momenta π_{q^i} . Thus, from the definition of the momenta and the Lagrangian $\mathcal{L} = \frac{\partial B}{\partial q^i} \dot{q}^i$, the primary constraints

$$\phi_i := \pi_{q^i} - \frac{\partial B}{\partial q^i} \approx 0. \quad (47)$$

arise. The canonical Hamiltonian $H_c = \dot{q}^i \pi_{q^i} - \mathcal{L}$ identically vanishes. On the other hand, the evolution of the primary constraints is strongly zero and so (47) are first-class. The Hamiltonian action principle acquires the form

$$S[q^i, \pi_{q^i}, u^i] = \int_{\tau_1}^{\tau_2} d\tau (\dot{q}^i \pi_{q^i} - \lambda^i \phi_i). \quad (48)$$

The dynamics of this theory is pure gauge in the sense that the number of physical degrees of freedom is zero, $\frac{1}{2}(2N - 2N) = 0$, and evolution in τ is the unfolding of the gauge symmetry.

By other hand, the boundary term which come from the gauge transformations generated by the first-class constraints is the following

$$M_\phi = \varepsilon \frac{\partial B}{\partial q^i}. \quad (49)$$

By one hand, this example is relevant because it illustrates the role played by boundary terms only. By other hand, this result can be expressed as follow

Theorem 1 *Let \mathcal{M} be a m -dimensional manifold with boundary $\partial\mathcal{M}$ of dimension $m-1$. Let $L_B(q^i, \dot{q}^i)$ and $L_F(q^i)$ be two Lagrangian functions defined on \mathcal{M} and $\partial\mathcal{M}$ respectively, where q^i are N coordinates which label the points of the configuration space. Therefore, if L_B such as*

$$L_B(q^i, \dot{q}^i) = \frac{d}{dt} L_F(q^i) \quad (50)$$

then the theory defined by L_B is topological.

The theorem is still valid if L_F depend on the momenta p^i or the higher n -th derivative of the coordinates[‡] $\frac{d^n q}{dt^n}$, namely, if the Lagrangian L_B can be expressed as the total derivative of some function L_F the number of the degrees of freedom will be zero, namely the theory always will be topological. This theorem is true for system with finite numbers of degrees of freedom and as well as for field theories as it is seen in the appendix and in [15].

4. Concluding remarks

It has been studied how to manage time boundary terms in the theoretical framework of Dirac's canonical analysis. The strategy consist in the introduction of the time boundary term into the action principle, thus enlarging the original set of configuration variables. The resulting action is interpreted as a Lagrangian one to which the canonical analysis can be applied. In this approach the time boundary conditions of the new action principle are on the new full set of configuration variables, as usual. The information of the boundary term is encoded in the new second-class constraints as well as in the redefinition of the new first-class constraints.

The approach followed here, can also be applied to the case when the original action principle is endowed with arbitrary symplectic structures instead of canonical ones [5].

It is worthwhile to mention that the boundary term added can also include the original Lagrange multipliers. The analysis in such a case can be carried out following essentially the same steps made in the current paper.

As final comments, the approach developed here can also be applied to mini-superspace models (when the space-time has specific symmetries), such as cosmological models [see, for instance, Ref. [13-19]. Moreover, these way to deal with boundary term could be applied into the zero-Hamiltonian problem in 2D gravity [20, 21] and into topological field theories [15]. Finally, the approach developed here can extended to its complex counterpart and analyze complex canonical transformations [22].

[‡] The demonstration of the theorem when L_F depends on the higher derivatives of the coordinates do not has been deployed here because it is out of the spirit of the paper.

Acknowledgments

This work was presented in the parallel session of the 12th Marcel Grossmann Meeting held in Paris France, 2009. We thank M Montesinos and JD Vergara for very fruitful discussions on the subject. This work was supported in part by CONACYT, Mexico, Grant No. 56159-F.

Appendix A. The Stokes's theorem as a topological field theory

In this appendix we will consider the Stokes theorem in two dimensions as special case of the theorem 1 for field theory. The Stokes theorem can be read as

$$\int_{\mathcal{M}^n} d\omega = \int_{\partial\mathcal{M}^n} \omega, \quad (\text{A.1})$$

where \mathcal{M}^n is a n -manifold with boundary, ω is an $(n-1)$ -form. For the sake of simplicity, let us consider a 2-manifold \mathcal{M}^2 . Let (x, y) be local coordinates that label the points of \mathcal{M}^2 . Therefore, $\omega = X(x, y)dx + Y(x, y)dy$. By one hand, the Stokes theorem (A.1) seen as an action principle acquires the form

$$S[X, Y] := \alpha \int_{\mathcal{M}^2} [\partial_x Y(x, y) - \partial_y X(x, y)], \quad (\text{A.2})$$

where α is the constant of proportionality which absorbs the unities. By other hand, we consider the parametrization of the coordinates (x, y)

$$x = x(\tau, \sigma), \quad \text{and} \quad y = y(\tau, \sigma). \quad (\text{A.3})$$

The range of the parameters are $\tau_1 \leq \tau \leq \tau_2$ and $\sigma_1 \leq \sigma \leq \sigma_2$, namely, we parameterize the surface \mathcal{M}^2 with a square. Solving the components of (A.2) in terms of the parameters (τ, σ) , we obtain

$$S[X, Y] := \alpha \int_{\mathcal{M}^2} d\tau \wedge d\sigma \left(\dot{X}x' + \dot{Y}y' - \dot{y}Y' - \dot{x}X' \right). \quad (\text{A.4})$$

where the dot and the apostrophe denote derivation with respect to τ and σ respectively.

In order to make the counting of degrees of freedom, the momenta $(\pi_i) = (\pi_X, \pi_Y, p_x, p_y)$ canonically associated to the coordinates $(q^i) = (X, Y, x, y)$ must be defined, and these arise four primary constraints

$$\begin{aligned} \phi_X &:= \pi_X - \alpha x' \approx 0, & \phi_Y &:= \pi_Y - \alpha y' \approx 0, \\ \phi_x &:= p_x + \alpha X' \approx 0, & \phi_y &:= p_y + \alpha Y' \approx 0. \end{aligned} \quad (\text{A.5})$$

A straightforward computation implies that the canonical Hamiltonian vanishes, and so the action principle acquires the form

$$S[q^i, \pi_i, \lambda^i] = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1}^{\sigma_2} d\sigma \left(\dot{q}^i p_i - \lambda^i \phi_i \right) \quad i = 1, \dots, 4. \quad (\text{A.6})$$

By consistency, the primary constraints (A.5) must be evolved respect to the parameter τ . These τ -evolution are strongly zero and therefore there are no more constraints. Moreover, the algebra of constraints tell us that the constraints (A.5) are first-class.

The extended phase space is parameterized by 4 configuration variables q^i and the corresponding 4 canonical momenta π_i , there are 4 first-class and 0 second-class constraints. Therefore the system has $\frac{1}{2}(2 \times 4 - 2 \times 2) = 0$ physical degree of freedom

per point of σ , namely, the theory defined by the Stokes theorem is topological as was pointed out in the theorem 1.

As final comment it is worth noticing that besides the left side of the Stokes theorem (A.1) is topological, the canonical analysis of the right side of (A.1) reveals that the theory defined by $L_F = \alpha\omega$ has one degree of freedom [14]. Namely, the theory described by (A.2) is other example of the theories defined in a manifold with boundary which are topological in the bulk and has local degrees of freedom into its boundary [15].

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